THE

MATHEMATICAL GAZETTE.

EDITED BY

W. J. GREENSTREET, M.A.

WITH THE CO-OPERATION OF

F. S. MACAULAY, M.A., D.Sc.; PROF. H. W. LLOYD-TANNER, M.A., F.R.S.; E. T. WHITTAKER, M.A.

LONDON:

GEORGE BELL & SONS, YORK ST., COVENT GARDEN,

AND BOMBAY.

ON VON STAUDT'S GEOMETRIE DER LAGE.

(*Continued from page* 314.)

HAVING adjoined the ideal elements to the visible universe, von Staudt proves the usual theorems of projective geometry, noting the different cases that arise from different arrangements of the elements. As the only means of comparing figures is by projection, all the proofs depend on this; in many cases on the theorem that a given cast (*Wurf*) is unaltered when any two elements are interchanged, provided the other two are interchanged also (*G.* 59), which can be written symbolically

$$(ABCD)=(BADC)=(CDAB)=(DCBA),$$

while if the cast is harmonic, viz. AC harmonic with respect to BD, $(ABCD)=(CBAD)=(ADCB)$. The arrangement is very much that followed by Reye (*Geometrie der Lage*; see also Henrici, *Encyc. Brit.*, Art. "Geometry," Sect. II.), except in the first treatment of curves and surfaces of the second order, which is made to depend on a very general view of involution, a conception much more fundamental in von Staudt than in any of his followers.

If two projective forms of the same nature (*e.g.* ranges) lie in the same base, they may lie in such a manner as to pair the elements of the base. This does not necessarily occur; the point of the first range that lies at any point A of the base has in the second range a correspondent that lies at A'; there is also a point of the second range at A, and the corresponding point of the first range may happen to be, not at A', but at a different point A''. If however A'' is the same as A', so that the points A A' are definitely associated, it follows that any other point B has a definite associate. For let the two correspondents to B be B' and B'', then the four points, A, A', B, B'', regarded as belonging to the first range, have for correspondents in the second range

A', A, B', B. As the ranges are projective, these two casts are projectively equivalent, and by permissible interchanges the second is equivalent to $(AA'BB')$. We have therefore

$$(AA'BB'') = (AA'BB'),$$

which shows that B' and B'' are the same point. Thus every point of the base has a definite associate; by means of the two projective ranges the points of the base are paired. The two ranges are said to be in involution.

This arrangement, however, is truly apprehended only when looked upon as an association of the elements of one base (*G.* 120, *B.* 44); and while von Staudt does introduce it in the way just mentioned, namely, by means of superposed projective forms (the most natural way from his point of view, that figures are to be judged by their appearance), yet he immediately and permanently lays stress on the "projective pairing" of the elements of the common base.

By means of two pairs of the involution the point associated with any arbitrary point can be found, for example, by means of the three pairs of sides of a complete quadrangle (*G.* 122). If AA', BB' are two pairs of points in an involution, the segment ABA' corresponds to the segment $A'B'A$; hence the correspondent to any point in ABA' lies in $A'B'A$, and it is consequently impossible for a point to coincide with its correspondent (correspond to itself) unless these segments are the same, that is, unless B and B' lie in the same segment AA'. If this is the case, as a point describes ABA', its correspondent describes the same segment in the reverse order $A'B'A$; hence at some point in this segment the two coincide. As the same argument applies to the remaining segment AA', there is another self-corresponding point. Thus if corresponding elements are separated by corresponding elements, there are no self-corresponding elements (double elements, *Ordnungselemente*); while if corresponding elements are not separated by corresponding elements, there are two double elements. Plainly if there is known to be one such element, there is also a second. There cannot be more than two, as the coincidence of three elements of two projective forms on the same base makes the forms identical. Calling these F_1, F_2, the four elements A, A', F_1, F_2, correspond to A', A, F_1, F_2; hence $(AA'F_1F_2) = (A'AF_1F_2)$, that is, F_1F_2 are harmonic with respect to AA', and thus with respect to any pair in the involution.

The nature of an involution is determined by two pairs of elements; if the two elements of one pair are *not* separated by those of the other, the involution has two double elements, and is now called hyperbolic; if the two elements of one pair *are* separated by those of the other, the involution has no double

elements, and is called elliptic. These names are not used by von Staudt.

The first conception of involution—namely, that it is a particular projective relation of two forms—is applicable to projective forms whose elements are not of the same nature, for example, to a flat pencil and a range in its plane. The flat pencil determines on the base of the range a second range, projective with the first, and if the two ranges are in involution, the given range and flat pencil are said to be in involution. Any extensive use of this application would obscure the true inwardness of an involution, for the given elements are not even in one base, and accordingly von Staudt does not employ it much in the case of primitive forms of the first grade. He makes more use of it as regards primitive forms of the second grade. In the plane, for instance, the association of pairs of points AA', etc. will make the line joining two of a pair correspond to itself; the correspondence can be shown to be a highly specialised perspective relation, known as harmonic projection, which, though sufficiently interesting in itself, has not any very exciting consequences. But if the points and lines are made to correspond projectively to the lines and points, in such a way that a point P is definitely associated with a line p, we have a relation of the nature of involution; it does consist in the projective pairing of elements of one form, only now paired elements are unlike in nature. This association is indicated by the name polar; the plane is regarded as a polar system, the associated point and line being pole and polar (*G.* 131).

If B is any point on a, then b, the polar of B, passes through A, the pole of a. This is an immediate consequence of the fact that the range a corresponds projectively to the pencil A. Two points such that each lies on the polar of the other, or two lines such that each passes through the pole of the other, are said to be conjugate. If a point lies on its own polar, it is conjugate to itself; likewise the line is conjugate to itself.

Let b meet a in B'; then b', the polar of B', passes through B. Hence, on any line a, pairs of conjugate points form an involution; and pairs of conjugate lines through a point A form an involution. The triangle ABB' has the property that each vertex is the pole of the opposite side; it is called a polar triangle.

If a point P lies on its polar p, and Q is any other point on p, since the polar of Q passes through P, and is not the line p, it follows that the polar of Q cannot pass through Q. Call it q, and consider the involution of conjugate points on q. This has certainly one self-corresponding point, P, hence it has one more; that is, on any line through P, other than p, there is one point R, distinct from P, that lies on its own polar r. By the same

argument, applied to the pencil through Q, it is seen that through any point on p, other than P, there is one line r, distinct from p, that passes through its own pole R. Hence if in a plane polar system there is one point that lies on its own polar, there is a locus of such points, which has the property that, with one exception, any line through one of its points meets it in one more. The one exception is the polar of P, which meets the curve at P only, and is therefore a tangent. Similarly there is a system of lines that pass through their own poles, and with one exception, through any point on one of these lines there passes one more. The one exception is the pole of p, which is therefore the point of contact; and what has just been said shows that the locus of the points, and the envelope of the lines, is one and the same curve. This curve, which has the properties that if a line other than a tangent meets it at all, it meets it in two points, and if from a point not on the curve a tangent can be drawn, there are two such tangents, is called a conic.

The two points in which a line meets a conic have been shown to be the double points of the involution of conjugate points on the line. Let the chord AB through Q meet q in Q', then QQ' are conjugate points, hence a pair in the involution of which AB are double points. The chord AB is therefore divided harmonically by Q and q; and taking different chords through Q, the locus of the harmonic conjugates to Q is the line q. This applies whether the line q cuts the curve or not; if it does cut the curve in P, R, the tangents at P, R are the lines p, r, which necessarily pass through the pole of PR, that is, through Q; hence the polar of Q is the chord of contact of tangents from Q. From this all the usual polar properties with reference to the conic follow at once.

The facts that the points of a conic are the intersections of corresponding rays of projective pencils, and the tangents the lines joining corresponding points of projective ranges, which are now usually taken as the starting-point in dealing with the curves, are proved by the consideration of certain forms in involution (*G.* 142). It seems to me that von Staudt's treatment is at once simpler, and more pleasing in the breadth of its conception. He treats the curve as a whole, and thus does not find it necessary to prove that all points of it have the same properties. Moreover, his conception brings out clearly the self-reciprocal nature of the curve, laying stress equally on the points and lines; thus it is natural that all the polar relations, which are essentially self-reciprocal, should be more simply derived by von Staudt's plan than by the modified form adopted by Reye.

The cone is discovered in a sheaf in precisely the same manner as the conic in a plane; surfaces of the second order are obtained

by associating the points and planes of space with the planes and points, so as to obtain a projective pairing of elements of a different nature; in this a line regarded as the base of a range is associated with a line regarded as the base of an axial pencil (*G.* 190-203).

By means of a flat pencil whose vertex is on the conic, the curve is viewed as in perspective with the pencil, and with the range determined by the pencil on any transversal. Thus we may consider the range of points on a conic, and this can be made to be projective with a range on a straight line, or on another conic, or on the same conic. Consequently the conception of involution is applicable to a conic; and it is shown that when the points of a conic are projectively paired, the pairing is necessarily perspective—the lines joining the elements of the pairs all pass through one point, the pole of the involution. There are, or are not, self-corresponding elements according as tangents can, or cannot, be drawn through this point; hence the self-corresponding elements are the intersections of the curve and the polar of this point, and can be constructed at once when the conic and two pairs of the involution on it are given (*G.* 139, 158-161, *B.* 47). The involution is therefore elliptic or hyperbolic according as its pole is inside the conic or outside the conic.

As an involution on a line can be projected on to a conic, by means of a perspective pencil whose vertex is on the curve, and as the double points of the one involution then project into those of the other, it follows that the double elements of an involution can be constructed if a single conic is given. The common pairs of two involutions on the same base can similarly be constructed; for projecting the two involutions on to the one conic, through the same point as vertex, let the two poles be O_1, O_2; then as the elements of any pair in an involution lie on a line through the pole, the only common pair is that in which the line O_1O_2 meets the conic, and if this line does not meet the conic, there is no common pair. Plainly the line does meet the conic if either point O is internal; and may meet it even if both points O are external. Hence two involutions on a common base have one common pair of elements, certainly if one involution is elliptic, possibly if neither is elliptic (*G.* 178, 182).

Towards the end of the *Geometrie* (p. 186) von Staudt points out that a theorem apparently depending on the double elements of an involution can generally be stated without reference to these, and that when thus stated it will hold even for an involution that has no double elements. Thus, for example, instead of saying that a conic meets a line in the double points of a certain involution AA', BB', we can say that AA', BB', are pairs of conjugate points with reference to the conic. With this statement it is immaterial what the nature of the involution may be;

two conditions are in any case imposed on the conic. Thus there is no *necessary* difference in statement according as an involution has or has not double elements; and to enable us to bear this in mind, we speak of double elements in both cases, qualifying them in the one case as feigned or imaginary. This is a mere *façon de parler*; and anyone that prefers can dispense with it (p. 187). Not a very dogmatic way of introducing imaginary elements! and very unlike von Staudt's usual style; but the explanation is to be found in the preface to the *Beiträge*, written nine years later, where he says that he could not at that time enter more minutely into the theory of imaginary elements, because he had not evolved any way of distinguishing between the two of a pair. Thus his mind was full of the question, and he could not resist mentioning it, though he was not then prepared to enlarge upon it, nor, I think, did he desire to direct attention too emphatically to it.

The difficulty was this: his geometry accepts and deals with elements singly, not as pairs; but the involution AA', BB', determines a pair. When the involution is hyperbolic, the elements can be marked; we point at them, and attach names to them, saying *this* is X and *that* is Y; or even without the names, they are sufficiently distinguished by their place, we can specify in which of the two segments AA' each lies. But if the involution is elliptic, we cannot indicate the double elements in any graphic manner; we cannot point at them, nor can we say that either one lies in any specified segment of the base. It is necessary therefore to find some distinguishing geometrical property of the two points, some geometrical concept other than that of position, that may naturally be associated with them in such a way as to distinguish them. So far as the mere naming of the points is concerned, there is no difficulty; we may call them black and white, this and that, first and second, anything except the proverbial Irish B for before and B for behind! we associate two names arbitrarily with the points as a means of referring to them. It is really analogous to assigning arbitrarily chosen distinguishing marks to practically indistinguishable twin babies—it is absolutely indifferent in the first instance which has which, only they must keep to them.

Von Staudt's distinction may perhaps be made clearer by some preliminary considerations. Suppose we know that imaginary elements are subject to the same laws of combination as real elements; that is, two lines in a plane determine a point, etc. For the present, we can speak of the two elements of one pair of imaginaries arbitrarily as first and second; but we cannot make the choice arbitrarily for every pair, we must arrange matters so that they shall not be disarranged by projection. If two forms, for example two ranges, are in perspective, then the first of

a pair in the second range must be the one that projects into the first of the corresponding pair in the first range. Now a pair of imaginary elements can be indicated by any two pairs of the determining involution. To get rid of the multiplicity of representation, we select some one, AA', BB'; and with a view to the projective comparison of different forms, it is convenient to choose this one so that the two pairs shall be harmonic. This can always be done; starting with any arbitrarily selected element A whose conjugate is A', we have to find in the given elliptic involution a pair that shall be harmonic to AA', that is, a pair that shall belong also to the hyperbolic involution whose double elements are AA'; and one such pair has been shown to exist. Hence the imaginary double elements of an elliptic involution can always be indicated by means of a harmonic cast, one of whose elements can be chosen arbitrarily; and in comparing, for example, two pairs of imaginary points on one line, we can represent both pairs by a harmonic cast starting from an arbitrary point A. Let the pairs (X), (Y) be given by $(ABA'B')$, $(ACA''C')$, these being written so as to represent the same sense on the line. Let either X be indicated arbitrarily as X_1; when $(ACA''C')$ is projected into $(ABA'B')$, one of the Y's comes at X_1, call this Y_1. Now without interfering with the selected starting-point A, we can project the second given cast into the first, taken in the reversed sense, namely, $(ACA''C')$ into $(AB'A'B)$; for harmonic properties show that

$$(AB'A'B)=(ABA'B')=(ACA''C').$$

It will now be Y_2 that comes at X_1. To prove this, project $(ACA''C')$ from a centre V on to any line v through A, distinct from the given base u, so obtaining $(Aba'b')$, which is necessarily in perspective with $(ABA'B')$ through some centre O, and with $(AB'A'B)$ through a different centre O', where the line OO' is $A'a'$. Let Y_1Y_2 become x_1x_2 on v, then these project from O into X_1X_2 on u; by projection from O' the two x's become the two X's, but as x_1X_1 is a line through O, distinct from OO', it follows that x_1O' cannot pass through X_1, and must therefore pass through X_2. Thus the two Y's can be discriminated by means of the sense associated with the determining cast. If we take this in the sense ABA', and compare it projectively with $(ABA'B')$, a particular Y corresponds to X_1, this is Y_1; if we take it in the opposite sense, it is the other Y, Y_2, that corresponds to X_1, hence we can specify this Y by associating with the determining cast the sense $A'BA$, that is, $AB'A'$.

This shows that if we can prove that the imaginary elements are subject to the same laws as real elements, then the discrimination between two of a pair can be effected by means of the sense of the form in which the elements lie. A more precise

geometrical interpretation of this association will be given later; at present it must be accepted as it is given by von Staudt, as a somewhat arbitrary means of distinguishing the two elements, with an *a posteriori* justification.

Our domain so far consists of the visible universe together with one ideal plane, containing ideal points and lines. We now explicitly adjoin to this imaginary elements, determined in pairs of conjugates as the double elements of elliptic involutions, and discriminated by associating with the involution the idea of sense (*B.* 76, 77) These new elements are

(1) imaginary points, lying by pairs on real lines, all on any one line being divided into two classes by the association of sense with the involution on the line, conjugates being in different classes: the division of one pair on each real line is in general arbitrary;

(2) imaginary planes, similarly derived in pairs from an axial pencil with a real base;

(3) imaginary lines of the first kind, existing in a flat pencil.

There is still another kind of imaginary line, existing by pairs on the regulus determined by two projective ranges whose bases are not coplanar. Let the harmonic cast $(ABA'B')$ on the one line p correspond to $(GHG'H')$ on the other line q, then the imaginary points X_1, X_2 on p correspond to the imaginary points Y_1, Y_2 on q. The lines X_1Y_1, X_2Y_2 are conjugate imaginary lines determined by the skew involution aa', bb', where a is AG, etc. Now these lines do not intersect, for if they did, p, q would lie in their plane; consequently they do not exist in a flat pencil. Hence the elements adjoined must include

(4) imaginary lines of the second kind, existing in a real regulus.

Thus every one of the four primitive forms of the first grade yields its own kind of imaginary element. Von Staudt states expressly that there are no other imaginary elements; that is to say, the admissible operations, performed on any of the elements already accepted, actual, ideal, or imaginary, will result in these elements only. The proof of this statement requires the detailed consideration of every operation on every possible set of elements (*B.* 78-83), which depends on the precise definition of the phrases "lie in" and "pass through" as applied to imaginary elements. An imaginary element lies in (or passes through) a real base of a different nature if the two pairs of determining (real) elements lie in the real base; hence not only the one imaginary element, but also its conjugate, will lie in the real base. An imaginary point lies in an imaginary plane if the real base of the range coincides with the real base of the axial pencil, and then both conjugate points lie in both conjugate planes; *or* if the range is in perspective with the axial pencil, and the *right one* of each pair of imaginaries is chosen—as von Staudt expresses it, if

the range AA', BB' and the axial pencil pp', qq', are in perspective, and the sense ABA' agrees with the sense pqp'. In this case the conjugate point lies in the conjugate plane. It follows that an imaginary point (or plane) and a real line determine an imaginary plane (or point).

Similarly every combination of an imaginary line of either kind with a point or plane is discussed, and it is found that the epithet imaginary does not affect the laws of combination. It appears too that if any combination of elements yields a certain result, then the same combination of the conjugate elements yields the conjugate result, where a real element must be regarded as conjugate to itself. This may be expressed in the form :—Conjugate operations applied to conjugate elements yield conjugate results.

These considerations prove von Staudt's assertion that the admissible operations do not lead us outside the domain last defined; the elements accepted form the complete system. The laws expressing the possibilities of this complete system of elements are then restated (any two points determine a line, etc., *B.* 83-86), and it is pointed out that the greater part of the theorems proved in the *Geometrie der Lage* apply whether the elements are real or imaginary.

C. A. SCOTT.

(*To be continued.*)

EXAMPLE OF TRIGONOMETRICAL PORISMS.

Mr. R. F. Davis has shown (*Gazette*, Oct., 1899) that the porismatic equations

$$\begin{aligned} &u\cos\beta\cos\gamma+v\sin\beta\sin\gamma+w+u'(\sin\beta+\sin\gamma)\\ &\qquad\qquad+v'(\cos\beta+\cos\gamma)+w'\sin(\beta+\gamma)=0,\\ &u\cos\gamma\cos\alpha+\ \ldots \qquad\qquad =0,\\ &u\cos\alpha\cos\beta+\ \ldots \qquad\qquad =0 \end{aligned}$$

lead to the porismatic condition

$$u'^2+v'^2-w'^2=w(u+v)-uv,$$

and to the symmetrical relations between α, β, γ,

$$\begin{aligned} (u+v)C_1-(u-v)C_3-2w'S_3+2v'=0,\\ (u+v)S_1-(u-v)S_3+2w'C_3+2u'=0, \end{aligned}$$

where $C_1=\Sigma\cos\alpha$, $C_2=\Sigma\cos(\beta+\gamma)$, $C_3=\Sigma\cos(\alpha+\beta+\gamma)$, with a similar interpretation for S_1, S_2, S_3 in the sines.

Mr. Davis gives four more forms of the symmetrical relations, which can be deduced from the above by the identities

$$C_2=C_1C_3+S_1S_3, \qquad S_2=C_1S_3-S_1C_3, \qquad C_3^2+S_3^2=1.$$

I had occasion to apply this theory (before the publication of Mr. Davis's note) to solve a certain geometrical problem which

gives a neat illustration of the results. The problem relates to the porism of triangles in- and circum-scribed to concentric coaxial conics.

Take a triangle inscribed in the conic $x^2/a^2+y^2/b^2=1$, and circumscribed to $x^2/a_1^2+y^2/b_1^2=1$.

Let α, β, γ be the eccentric angles of the vertices. Then the chord joining β, γ is

$$\frac{x}{a}\cos\frac{\beta+\gamma}{2}+\frac{y}{b}\sin\frac{\beta+\gamma}{2}=\cos\frac{\beta-\gamma}{2}.$$

This touches the second conic if

$$\frac{a_1^2}{a^2}\cos^2\frac{\beta+\gamma}{2}+\frac{b_1^2}{b^2}\sin^2\frac{\beta+\gamma}{2}=\cos^2\frac{\beta-\gamma}{2},$$

or if

$$\left(-\frac{a_1^2}{a^2}+\frac{b_1^2}{b^2}+1\right)\cos\beta\cos\gamma+\left(\frac{a_1^2}{a^2}-\frac{b_1^2}{b^2}+1\right)\sin\beta\sin\gamma$$
$$=\left(\frac{a_1^2}{a^2}+\frac{b_1^2}{b^2}-1\right).$$

So we have our poristic set of equations with

$$u=-\frac{a_1^2}{a^2}+\frac{b_1^2}{b^2}+1,\quad v=\frac{a_1^2}{a^2}-\frac{b_1^2}{b^2}+1,\quad w=-\left(\frac{a_1^2}{a^2}+\frac{b_1^2}{b^2}-1\right),$$
$$u'=v'=w'=0.$$

The porismatic condition is readily found to reduce to

$$\left(\frac{a_1}{a}+\frac{b_1}{b}+1\right)\left(-\frac{a_1}{a}+\frac{b_1}{b}+1\right)\left(\frac{a_1}{a}-\frac{b_1}{b}+1\right)\left(\frac{a_1}{a}+\frac{b_1}{b}-1\right)=0.$$

Further, the symmetrical relations between α, β, γ are

$$C_1-\left(\frac{b_1^2}{b^2}-\frac{a_1^2}{a^2}\right)C_3=0,$$
$$S_1-\left(\frac{b_1^2}{b^2}-\frac{a_1^2}{a_2}\right)S_3=0.$$

The other relations contained in these are

$$S_2=0,$$
$$C_2=\frac{b_1^2}{b^2}-\frac{a_1^2}{a^2}.$$

From the relation $S_2=0$, or $\Sigma\sin(\beta+\gamma)=0$, it appears that the normals at α, β, γ are concurrent, a well-known theorem.

We can find easily the locus of the centroid of the triangle, for

$$x=\tfrac{1}{3}\Sigma a\cos\alpha=\tfrac{1}{3}aC_1=\tfrac{1}{3}a\left(\frac{b_1^2}{b^2}-\frac{a_1^2}{a^2}\right)C_3,$$

$$y=\tfrac{1}{3}\Sigma b\sin\alpha=\tfrac{1}{3}bS_1=\tfrac{1}{3}b\left(\frac{b_1^2}{b^2}-\frac{a_1^2}{a^2}\right)S_3.$$

Hence the locus is $\frac{x^2}{a^2}+\frac{y^2}{b^2}=\frac{1}{9}\left(\frac{b_1^2}{b^2}-\frac{a_1^2}{a^2}\right)^2$,

an ellipse similar to the outer of the given ellipses.

Another application will be to find the point of intersection of normals at α, β, γ. It may readily be shown that this is in general given by

$$\frac{2ax_0}{a^2-b^2}=C_1-C_3=\left(\frac{b_1^2}{b^2}-\frac{a_1^2}{a^2}-1\right)C_3 \text{ here,}$$

$$\frac{2by_0}{a^2-b^2}=-S_1-S_3=\left(\frac{a_1^2}{a^2}-\frac{b_1^2}{b^2}-1\right)S_3 \text{ here.}$$

These can be further reduced if we suppose the porismatic condition to be $\frac{a_1}{a}+\frac{b_1}{b}=1$, as is usually the case. Then we find

$$x_0=-\frac{a_1}{a^2}(a^2-b^2)C_3,$$

$$y_0=-\frac{b_1}{b^2}(a^2-b^2)S_3.$$

It may now be verified that with this combination of signs $\left(\frac{a_1}{a}+\frac{b_1}{b}=1\right)$ the side β, γ is

$$\frac{x}{a_1}\cos\alpha+\frac{y}{b_1}\sin\alpha+1=0,$$

or is the polar of the vertex α with respect to

$$\frac{x^2}{aa_1}+\frac{y^2}{bb_1}+1=0.$$

Hence all the poristic triangles are self-polar with respect to this conic.

T. J. I'A BROMWICH.

REVIEWS.

Exercices d'Arithmétique. Enoncés et Solutions. Par J. FITZPATRICK, Ancien Professeur de Mathématiques, et GEORGES CHEVREL. 2me édition. Paris, Hermann, 1900; 8vo, pp. 680.

In England Arithmetic has always been the Cinderella of the Sciences. It has been her portion to do the marketing, to sweep up after Mensuration, to set the rooms of Physics in order, and to be a nursemaid to Algebra. She has never had a room of her own. It is only in France that she may put on gala-dress and appear as a princess in her own right. In the Germany where Gauss wrote she is almost forgotten, and it is doubtful whether she was ever known in England. Nor must English readers of the delightful volume named above expect to find a quite familiar figure therein. It is Arithmetic indeed, but most of it would receive short shrift at the hands of those terrible English ex-

aminers who head their papers, "No question will receive any credit where other than Arithmetical Methods are employed." It is Arithmetic as Gauss knew it, but it is not one of the three R's; local examiners know it not, and it is a stranger unto the Department.

It is, to say the least, doubtful whether one English teacher in ten knows what Arithmetic is, and where Arithmetic stops and Algebra begins. To take a concrete example. Your problem is, "An article bought for a certain sum is sold at 27s., which is one-eighth as much again as the cost-price. What was the cost-price?" If you solve this thus:

$$\text{“}x+\frac{x}{8}=27, \quad \therefore \ \frac{9x}{8}=27, \quad \therefore \ x=24,\text{”}$$

your solution is ruthlessly crossed out as "Algebra." Write it

$$\text{“Cost-price} + \tfrac{1}{8}(\text{cost-price}) = 27,$$
$$\therefore \ \tfrac{9}{8} \text{ cost-price} = 27, \quad \therefore \text{ cost-price} = 24,\text{”}$$

and you will probably get full marks for your arithmetical ability. This means that if you denote the unknown by one letter (x) it is Algebra, if by nine letters (cost-price) it is Arithmetic. Could anything be more childish? Or, again, it is Arithmetic, say text-books and examiners, to write

$$\text{“Simple interest} = \frac{\text{Principal} \times \text{rate} \times \text{number of years}}{100},\text{”}$$

but it is Algebra to abbreviate it into $\frac{Prn}{100}$. So that it is clear that to most people Arithmetic is Algebra writ large, and the best arithmetician is the most garrulous algebraist. The real distinction between Arithmetic and Algebra, which appears to us to lie in the fact that the second is the science of continuously varying quantity, of number regarded as capable of changing from $+\infty$ to $-\infty$, is altogether overlooked, and stress is laid upon the accident of denotation of a number by a particular kind of shorthand. The necessary result has followed, and Arithmetic has become a farrago of rules whose value is measured by their commercial utility, while Elementary Algebra has, in our judgment, entirely lost its educational value.

The present book is a protest against such a lowering of ideals. Arithmetic to our authors is the Science of Numbers, and includes the greater part of the Elementary Theory of Numbers. It is true that in the present edition they have bowed the knee to Baal, and "added a large number of exercises of commercial arithmetic"; but as the answers to these are not given, nobody will be likely to use this part of their book, and there is no harm done. The rest of the book is full of interest. It often reminds one strongly of the brilliant work of M. Edouard Lucas; and whole pages are quoted from the *Récréations Mathématiques*, the *Théorie des Nombres*, and that fascinating fragment *L'Arithmétique Amusante*; but it is no mere copy and here and there is to be found a suggestive novelty. The authors, it must be said, lack the firm touch of M. Lucas. One can never feel quite safe in their hands. We rubbed our eyes, uncertain whether we saw aright, when on page 129 we read,

"In order that an odd number p should be prime, it is necessary and sufficient that the square of the number $\frac{p-1}{2}$ added to p should give a perfect square."

Had the problem of the ages, we wondered, been solved at last, and was the solution so childishly simple? We were disillusioned on page 501, where the authors confess their blunder and delete the word "sufficient." But the blunder never ought to have been made. Even less happy is the foolish Question 403: "Prove that, if $\phi(n)$ is a multiple of $n-1$, the number n is prime," where $\phi(n)$ is the number of numbers less than n and prime to it.

Occasionally, too, the solutions when correct are far from elegant. We may instance Question 209: "Prove that there exists no whole number which, substituted for n in $\frac{n-6}{15}$ and $\frac{n-5}{24}$, makes both fractions integral." The authors fail to notice that this problem is equivalent to the obvious "prove that $n-6$ and $n-5$ cannot both be divisible by 3." Moreover, we question whether the widest interpretation of the term Arithmetic can include the theory of the balance; and when our familiar tradesman with his fraudulent balance enters in Question 260 we think that occasion has been given to the adversary, even though the question be elegant: "Prove that a tradesman who uses the pans of his balance alternately for weighing his goods is bound to lose."

But with all its defects, and they are not few, the book is suggestive and of sterling value. It is just the kind of work to stimulate original thought in a boy with the elements of Algebra at command, and the not uncommon conviction that he "knows Arithmetic." It will show him refracted vistas beyond his horizon, confusedly perhaps, but none the less temptingly. The present writer may be pardoned a personal reminiscence. He regards the most important stimulus to his own mathematical development to have been the action of a master who put into his hands Whitworth's *Choice and Chance* when he was just beginning Algebra. Such a question as "Show that in taking a handful of shot from a bag it is more probable that an odd number will be drawn than an even number," is eminently provocative of thought, despite its faulty English. There are many such questions in the book before us. For instance, there is an elegant application of the binary scale of notation in a method of multiplication, said to be actually employed by Russian peasants, which it will interest our readers to see.

"Russian peasants, when they have to make a multiplication, usually proceed as follows: Take for example the multiplication of 35 by 42. They divide the multiplicand by 2 and at the same time double the multiplier, and, if the multiplicand is odd, they mark the multiplier by the sign +. Thus, in the given example, they would write:

35	42+
17	84+
8	168
4	336
2	672
1	1344+

Adding together the numbers marked +, they obtain the product 1470."

Or, again, there are the frequent hints, always so fascinating to a boy, that in certain provinces of Arithmetic we are literally at the outposts of human reason, and that until the great army has pushed its conquests further we must stand "with a wild surmise Silent upon a peak in Darien." Such are the references to Fermat's work, and many of the quotations from Lucas. For instance, on page 158: "Father J. Pervouchine, of Perm, has verified that the number $2^{2^{23}}$ composed of 2525223 figures is divisible by the prime number 167772161." Or, on the same page, the collection of formulae, from Lucas, representing prime numbers, though, of course, not exclusively:

$$x^2+x+41, \quad x^2+x+17, \quad 2x^2+29 \text{ (Euler)}.$$

Starting from $x=0$, these formulae give prime numbers for all values of x up to 40, 16, and 28 respectively. As our readers will be aware, no one has yet disproved Lucas' conjecture that the numbers of the series

$$2+1, \quad 2^2+1, \quad 2^{2^2}+1, \quad 2^{2^{2^2}}+1, \ldots \text{ are primes.}$$

On page 366 we find, quoted from Lucas' *Récréations*, the remarkable result of M. Le Lasseur concerning the factorisation of $a^{1260}-b^{1260}$, where a and b are the roots of $x^2=x-2$. The last five of the factors given, all of which are asserted to be prime, are 125541359, 25215201901, 34449677641, 153790567559, 733268745721. Or, once more, on page 8 we find the solution of a pretty question which our readers may care to try for themselves:

"The numbers 123456789101112 ... being written continuously without any stops between them, what is the nth digit of the series?"

We commend the book to all who have the true interests of Arithmetic at heart, and we cannot do better than quote the suggestive words from the preface contributed by M. Jules Tannery: "The problems of Arithmetic and of Elementary Geometry have a special value for the formation of the mind and the development of initiative. Each of them is something fresh. Ingenuity is needed to discover in the new problem attacked any feature of resemblance to those already known, and it is in the art of recognising these hidden resemblances, these distinct affinities, that mathematical talent lies." Only be it understood that by "the problems of Arithmetic" M. Tannery does not primarily mean the proceedings of stockbrokers. W. P. WORKMAN.

MATHEMATICAL NOTES.

79. *Note on the Trilinear Coordinates of the Focoids.*

[Read before the Mathematical Association, Saturday, January 27th, 1900.]

If ABC be the triangle of reference, the trilinear coordinates of the focoids (imaginary circular points at infinity) are such that for one

$$-\beta/\gamma=e^{iA}; \quad -\gamma/\alpha=e^{iB}; \quad -\alpha/\beta=e^{iC};$$

and for the other

$$-\gamma/\beta=e^{iA}; \quad -\alpha/\gamma=e^{iB}; \quad -\beta/\alpha=e^{iC}.$$

[NOTE. $-1=\cos\pi+i\sin\pi$.]

The above coordinates satisfy the equation of the line at infinity

$$\Sigma\alpha \,.\, \sin A=0,$$

for substituting the coordinates of the first focoid, we have

$$\sin A - \sin B(\cos C - i \sin C) - \sin C(\cos B + i \sin B) = 0,$$

identically.

They also satisfy the equation to the circum-circle

$$\sin A/a + \ldots + \ldots = 0,$$

as may be shown in a similar manner.

The use to which these results may be put can be illustrated.

I. To find the conditions that the general trilinear equation of the second degree

$$\Sigma u\alpha^2 + \ldots + \Sigma 2u'\beta\gamma + \ldots = 0$$

should represent a circle.

This curve passes through a focoid if

$$u + ve^{-2iC} + we^{2iB} + 2u'e^{-iC}e^{iB} - 2v'e^{iB} - 2w'e^{-iC} = 0\ ;$$

whence

$$\left.\begin{aligned} u + v\cos 2C + w\cos 2B + 2u'\cos(B-C) - 2v'\cos B - 2w'\cos C = 0, \\ -v\sin 2C + w\sin 2B + 2u'\sin(B-C) - 2v'\sin B + 2w'\sin C = 0, \end{aligned}\right\}$$

from which the well-known symmetrical forms can be obtained.

II. To find the foci of the ellipse

$$\sqrt{a\cos A} + \ldots + \ldots = 0,$$

which touches the sides of the triangle of reference at the feet of the perpendiculars.

The straight line $l\alpha + m\beta + n\gamma = 0$ touches the ellipse if

$$\Sigma \frac{\cos A}{l} = 0.$$

If this passes through a focoid,

$$l - m(\cos C - i \sin C) - n(\cos B + i \sin B) = 0.$$

If, therefore, we eliminate l, m, n from the three equations

$$\left.\begin{aligned} &\Sigma l\alpha = 0, \\ &l - m(\cos C - i\sin C) - n(\cos B + i \sin B) = 0, \\ &\Sigma mn\cos A = 0, \end{aligned}\right\}$$

and separately equate to zero the possible and impossible parts, we get the equations of two conics whose four points of intersection are the foci required.

The elimination is long, but not difficult. The result

$$\alpha^2 \cot A + \beta\gamma \operatorname{cosec} A = \beta^2 \cot B + \gamma\alpha \operatorname{cosec} B = \gamma^2 \cot C + \alpha\beta \operatorname{cosec} C$$

(which has been tested by Mr. M'Vicker) should perhaps have some geometrical interpretation.

This form of the trilinear coordinates of the focoids (also embodied in Question 14433 of the *Educational Times* for January, 1900) was arrived at on my part without any idea of having been forestalled elsewhere. It was consequently very disappointing to find that in a recent edition of Salmon's *Higher Plane Curves* the late Professor Cayley gives in Chapter I. precisely the same thing: although to the best of my belief no illustration of its use is to be found in the same work.

I venture to think that the extension of trilinear coordinates so as to include complex quantities will prove instructive and interesting. For example, the trilinear coordinates of the minor foci of the Brocard Ellipse are e^{iA}, e^{iB}, e^{iC} and e^{-iA}, e^{-iB}, e^{-iC}; whence the equation of the line joining them

$$\alpha \sin(B \sim C) + \ldots + \ldots = 0,$$

and the various elements of the ellipse can be obtained.

R. F. Davis.

PROBLEMS.

350. [K. 2. a.] Every pair of rectangular Simson lines will intersect on the nine-point circle. Deduce the locus of the mid-point of the mid-point joins of the chords intercepted on any circle. Also show that the triangle formed by the Simson lines of A', B', C' with respect to a tringle ABC is similar to that triangle, if A', B', C' are the mid points of the arcs BC, CA, AB respectively. W. J. GREENSTREET.

351. [A. 1. b.] If m be the geometric mean of n quantities $a, b, \ldots k$, prove that

$$\Sigma\frac{(a-b)^2}{ab} \leqslant n\Sigma\frac{(a-m)^2}{am}$$

where the first summation extends to every pair of the quantities.
R. F. MUIRHEAD.

352. [K. 1. c.] Given three points A, B, C, and two parallel straight lines; find a point P on one of the lines, such that if PA, PB, PC cut the other line in α, β, γ, then shall $a\beta=\beta\gamma$. A. W. S.

353. [L¹. 17. e.] Two parallel tangents to a conic meet a variable bitangent circle in four points. Shew two of the connectors of these points pass through the foci of the conic, and that the remaining two are each equal to the focal axis of the conic, and are fixed in direction. J. A. THIRD.

354. [K. 11. a.] If the radii of two orthogonal circles with fixed centres vary, find the envelope of their common tangents. A. THOMPSON.

355. [K. 20. e.] a, b, c are the centres of the squares described externally on the sides BC, CA, AB of a triangle ABC. H is the orthocentre of the triangle. Tangents are drawn from a, b, c to the circles $HEAF, HFBD, HDCE$. If $t_r (r=1, 2, \ldots 8)$ be one of the tangents, $\Sigma t_r^2=4\Delta(2\cot\omega+3)$.
R. TUCKER.

356. [I. 17. a.] Find three consecutive numbers each of the form $\square+\square$. (Ex. $2248=42^2+22^2$; $2249=43^2+20^2$; $2250=45^2+15^2$.) I have six solutions with numbers under 1000, and eight with numbers between 1000 and 2000. Can anyone indicate more solutions between these limits?
W. ALLEN WHITWORTH.

357. [K. 11. a; 20. e.] A, B, C are the centres of three circles, radii r_1, r_2, r_3, and the radius of their orthogonal circle is ρ. Shew that

$$16\Delta^2\rho^2=\Sigma a^2r_1^4-2\Sigma r_2^2r_3^2bc\cos A-2abc\Sigma r_1^2a\cos A+a^2b^2c^2$$

where Δ, a, A etc. refer to the triangle ABC. A. C. L. WILKINSON.

358. [K. 1. c.] O, B, C, A are fixed collinear points; BD, CE are perpendiculars to OA. Find DE parallel to OA if OF is perpendicular to AE and equal to OD. R. B. WORTHINGTON.

359. [K. 20. c.] If the roots of $\tan\theta=m\tan(A+\theta)+n\tan(B+\theta)$ may be said to be α, β, γ, then

$$\sin(A-B).\Pi\sin\alpha=\sin A\,\Pi\sin(B+\alpha)-\sin B\,\Pi\sin(A+\alpha).$$

C. E. YOUNGMAN.

SOLUTIONS.

UNSOLVED QUESTIONS.—57, 129, 144, 152, 171, 175, 252, 271, 275, 279, 283, 285, 287, 306-8, 311-13, 320, 326-7, 334, 336-9.

Solutions of these questions, and of 340-359, should reach the Editor not later than April 15th. They will be published as space is available.

The question need not be re-written; the number should precede the solution. Figures should be very carefully drawn on a small scale, and on a separate sheet.

GLASGOW: PRINTED AT THE UNIVERSITY PRESS BY ROBERT MACLEHOSE AND CO.